# The performance of Yu and Hoff's confidence intervals for treatment means in a one-way layout


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## General framework used

1. Suppose that we have a "full" model which is assumed to adequately approximate reality for the purpose of constructing a frequentist confidence set for a specified vector parameter of interest.

   Prior information about the values of the parameters in the full model may result from previous experience with similar data sets and/or expert opinion and scientific background.

2. We seek a confidence set for the parameter vector of interest, with specified minimum coverage probability $1 - \alpha$, that utilizes this uncertain prior information, in the sense that it satisfies the following minimal requirement:

   This confidence set has a small expected volume relative to the expected volume of the usual confidence set, based on the full model and having coverage probability $1 - \alpha$, in those parts of the parameter space that the uncertain prior information tells us are more likely.

## Outline of the remainder of the talk

A. Tail method confidence intervals for a scalar parameter of interest $\theta$ are obtained by a simple extension of the usual equi-tailed confidence interval based on a pivotal quantity for $\theta$.

   It is obvious that these confidence intervals have the desired coverage probability $1 - \alpha$ throughout the parameter space.

   By an appropriate choice of the tail function, these intervals can be made to utilize uncertain prior information about $\theta$.

B1. Consider a balanced one-way layout for the comparison of $p$ treatments.

   Uncertain prior information that the treatment population means are equal.

B2. Yu, C. & Hoff, P. (2018) Adaptive multigroup confidence intervals with constant coverage. *Biometrika*.

Extend the tail method to obtain confidence intervals for the treatment population means that individually have specified coverage probability $1 - \alpha$ throughout the parameter space and utilize the uncertain prior information (i.e. satisfy the minimal requirement).

They assess the expected lengths of these confidence intervals using a semi-Bayesian analysis.

B3. Kabaila, P. (2024) On Yu and Hoff's confidence intervals for treatment means. *Statistics and Probability Letters*.

provides a revealing assessment of these expected lengths using a fully frequentist analysis.

## A. Tail method confidence interval for a scalar parameter of interest $\theta$

Suppose that the distribution of the random vector $\boldsymbol{X}$ is determined by $(\theta, \boldsymbol{\psi})$, where $\theta$ is the parameter of interest, whose possible values belong to the interval $\Theta \subset \mathbb{R}$, and $\boldsymbol{\psi}$ is a nuisance parameter vector.

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**Construction of an equi-tailed confidence interval for $\theta$ using a pivotal quantity**

Suppose that $g(\boldsymbol{X}, \theta)$ is a scalar function of $(\boldsymbol{X}, \theta)$ with a continuous distribution that does not depend on $(\theta, \boldsymbol{\psi})$.

In other words, $g(\boldsymbol{X}, \theta)$ is a pivotal quantity for $\theta$.

Let $F$ denote the cumulative distribution function of $g(\boldsymbol{X}, \theta)$.

Thus $F(g(\boldsymbol{X}, \theta))$ has a uniform distribution on the interval $(0, 1)$.

Hence

$$1 - \alpha = \mathrm{P}\big(\alpha/2 \le F(g(\boldsymbol{X}, \theta)) \le 1 - \alpha/2\big).$$

Suppose that $g(\boldsymbol{x}, \theta)$ is an increasing function of $\theta$, for each possible value $\boldsymbol{x}$ of $\boldsymbol{X}$.

Then
$$\left\{\theta : \alpha/2 \leq F\big(g(\boldsymbol{X}, \theta)\big) \leq 1 - \alpha/2\right\}$$
is a confidence interval for $\theta$, with coverage probability $1 - \alpha$.

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**Introduce the tail function $\tau(\theta)$**

As noted by Stein (1961), Bartholomew (1971) and Puza & O'Neill (2006), for any tail function $\tau : \Theta \to [0, 1]$,

$$C_\tau(\boldsymbol{X}) = \left\{\theta : \alpha\tau(\theta) \leq F\big(g(\boldsymbol{X}, \theta)\big) \leq 1 - \alpha + \alpha\tau(\theta)\right\}$$

is a confidence set for $\theta$, with coverage probability $1 - \alpha$.

The usual equi-tailed confidence interval for $\theta$ is obtained by setting the tail function $\tau(\theta) = 1/2$ for all $\theta \in \Theta$.

Suppose that the tail function $\tau(\theta)$ is a nonincreasing function of $\theta$. Then

$$C_\tau(\boldsymbol{X}) = \Big\{\theta : 0 \le F\big(g(\boldsymbol{X}, \theta)\big) - \alpha\tau(\theta) \le 1 - \alpha\Big\},$$

where $F\big(g(\boldsymbol{x}, \theta)\big) - \alpha\tau(\theta)$ is an increasing function of $\theta$, for each $\boldsymbol{x}$. Hence the confidence set $C_\tau(\boldsymbol{X})$ is an interval.

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Puza & O'Neill (2006) provide some insight into how the choice of tail function $\tau$ influences the expected length of the confidence interval for $\theta$.

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The big advantage of tail method confidence intervals is that the coverage probability constraint is effortlessly satisfied.

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The tail method is limited to the construction of a confidence interval for a scalar parameter $\theta$, for which we have a pivotal quantity, and which is required to utilize uncertain prior information about this same parameter $\theta$.

Prior information about a nuisance parameter cannot be utilized.

Disadvantages of tail method confidence intervals that are not the same as the usual equi-tailed confidence interval.

1. We would like any confidence interval for $\theta$ that utilizes the uncertain prior information to approach the usual $1-\alpha$ confidence interval for $\theta$, based on the full model, when the data and the prior information become increasingly discordant.

   The tail method confidence interval does not have this property.

2. Suppose that the set of possible values of $\theta$ is $\mathbb{R}$.

   Also suppose that $\tau(\theta)$ is an decreasing function of $\theta$ such that $\tau(\theta) \to 0$ as $\theta \to \infty$ and $\tau(\theta) \to 1$ as $\theta \to -\infty$.

   Then, typically, the expected length of the tail method confidence interval diverges to infinity, as the data and the prior information become increasingly discordant.

## B1. Uncertain prior information for a one-way layout for the comparison of treatments

For simplicity, consider a balanced one-way layout for the comparison of $p$ treatments. Assume homogeneous random error variances.

Suppose that for each treatment $j \in \{1, \ldots, p\}$, we have $n$ independent and identically $N(\theta_j, \sigma^2)$ responses denoted by $Y_{1j}, \ldots, Y_{nj}$. Here $\theta_1, \ldots, \theta_p$ and $\sigma^2$ are unknown parameters.

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As pointed out by the econometrician

Leamer, E.E. (1978) Specification Searches: Ad Hoc Inference with Nonexperimental Data. Wiley,

preliminary data-based model selection may be motivated by a desire to utilize uncertain prior information in subsequent inference.

It is very common to carry out a preliminary F-test of the null hypothesis that $\theta_1 = \theta_2 = \cdots = \theta_p$ against the alternative hypothesis that the $\theta_j$'s are not all equal.

This preliminary F-test may be motivated by the desire to utilize the uncertain prior information that $\theta_1, \theta_2 \ldots, \theta_p$ are equal or close to equal.

## B2. Yu, C. & Hoff, P. (2018) Adaptive multigroup confidence intervals with constant coverage. *Biometrika*

Made the novel observation that a tail function that is random and independent of the pivotal quantity for the parameter of interest still leads to a confidence interval with exactly the desired coverage.

Suppose that $\boldsymbol{X}$ and $W$ are independent random vectors.

Also suppose that, for every possible value $w$ of $W$, $\tau(\theta, w) : \Theta \to [0,1]$. So that $\tau(\theta, W)$ is a random tail function.

Then

$$\begin{aligned}
&\mathrm{P}\Big(\alpha\tau(\theta, W) \le F\big(g(\boldsymbol{X}, \theta)\big) \le 1-\alpha+\alpha\tau(\theta, W)\Big) \\
&= \mathrm{E}_W\Big(\mathrm{P}\Big(\alpha\tau(\theta, W) \le F\big(g(\boldsymbol{X}, \theta)\big) \le 1-\alpha+\alpha\tau(\theta, W) \,\Big|\, W\Big)\Big) \\
&= \mathrm{E}_W(1-\alpha) \\
&= 1-\alpha
\end{aligned}$$

Let $\boldsymbol{Y}_j = \left(Y_{1j}, \ldots, Y_{nj}\right)$ $(j = 1, \ldots, p)$.

Construct a confidence interval $\mathrm{CI}_1$ for $\theta_1$ based on a pivotal statistic $g(\boldsymbol{Y}_1, \theta_1)$ and random tail function $\tau(\theta_1, W)$, where $W$ estimates the spread of the population means $\theta_2, \ldots, \theta_p$ based on $\boldsymbol{Y}_2, \ldots \boldsymbol{Y}_p$.

Obviously, $\boldsymbol{Y}_1$ and $W$ are independent.
Hence $\mathrm{P}(\theta_1 \in \mathrm{CI}_1) = 1 - \alpha$. This coverage probability constraint is effortlessly satisfied.

The random tail function $\tau(\theta_1, W)$ is chosen such that the expected length of the confidence interval $\mathrm{CI}_1$ is relatively small when $\theta_2 = \cdots = \theta_p$.

Of course, the uncertain prior information that $\theta_1 = \theta_2 = \cdots = \theta_p$ implies the uncertain prior information that $\theta_2 = \cdots = \theta_p$.

Consequently, the confidence interval $\mathrm{CI}_1$ utilizes the uncertain prior information.

Using the same method of construction as $\mathrm{CI}_1$, construct confidence intervals $\mathrm{CI}_2, \ldots, \mathrm{CI}_p$ for $\theta_2, \ldots, \theta_p$, respectively.

Conclusion:
Confidence intervals $\mathrm{CI}_1, \ldots, \mathrm{CI}_p$ have been constructed such that $\mathrm{P}(\theta_j \in \mathrm{CI}_j) = 1 - \alpha$ ($j = 1, \ldots, p$) and they have the following expected length property.

The expected lengths of each of these confidence intervals are relatively small when $\theta_1 = \theta_2 \cdots = \cdots = \theta_p$.

In other words, these confidence intervals utilize the uncertain prior information that $\theta_1 = \theta_2 \cdots = \cdots = \theta_p$.

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To assess the expected lengths of $\mathrm{CI}_1, \ldots, \mathrm{CI}_p$, the authors place a prior distribution on $\theta_1, \ldots, \theta_p$ and average $\mathrm{E}(\text{length of } \mathrm{CI}_j)$ over this prior distribution for each $j$.

In other words, they assess the expected lengths of these confidence intervals using a semi-Bayesian analysis.

Kabaila, P. (2024) On Yu and Hoff's confidence intervals for treatment means. *Statistics and Probability Letters*

Provides a revealing assessment of expected length of the confidence interval $\mathrm{CI}_1$ using a fully frequentist analysis.

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For tractability, it is assumed that $\sigma^2$ is known.

This is equivalent to assuming that the number *n* of measurements of the response for each treatment is large.

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Assess the expected length performance of this confidence interval use the scaled expected length (SEL) defined to be

$$\frac{\mathrm{E}\big(\text{length of } \mathrm{CI}_1\big)}{\mathrm{E}\big(\text{length of the usual } \mathrm{CI} \text{ with coverage } 1-\alpha\big)}.$$

## SEL is a function of two scalar parameters $\xi$ and $\eta$

Let $\gamma_j = n^{1/2}\theta_j/\sigma$ $(j = 1, \ldots, p)$ and $\overline{\gamma}_{\text{-}} = \sum_{j=2}^{p} \gamma_j \big/ (p-1)$.

Now let $\xi = \gamma_1 - \overline{\gamma}_{\text{-}}$.
This parameter is a scaled measure of the difference between $\theta_1$ and the average of $\theta_2, \ldots, \theta_p$.

Finally let $\eta = \sum_{j=2}^{p} (\gamma_j - \overline{\gamma}_{\text{-}})^2$.
This nonnegative parameter is a scaled sample variance of $\theta_2, \ldots, \theta_p$.

$(\xi, \eta) = (0, 0)$ corresponds to all the treatment means being equal.

It is proved that the scaled expected length is a function of $(\xi, \eta)$, which we denote by $\mathrm{SEL}(\xi, \eta)$.

## Two theorems:

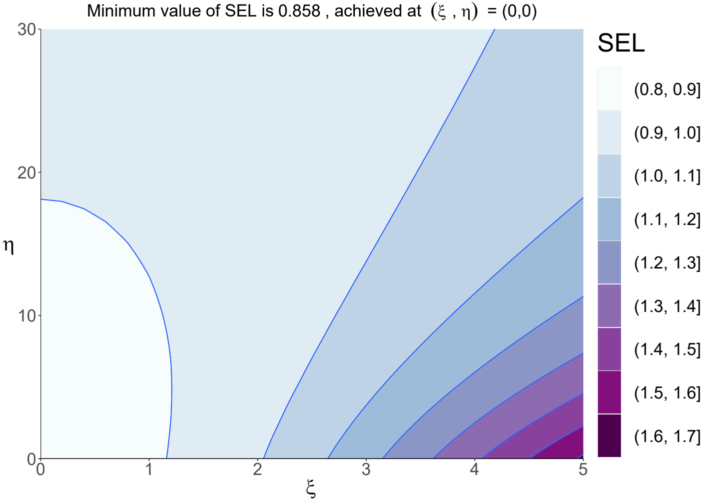
**Theorem** For every given $\eta$, $\mathrm{SEL}(\xi, \eta)$ diverges to $\infty$ as either $\xi \to \infty$ or $\xi \to -\infty$.

**Theorem** For every given $\xi$, $\mathrm{SEL}(\xi, \eta) \to 1$ as $\eta \to \infty$.

The following is a contour plot of $\mathsf{SEL}(\xi, \eta)$ as a function of $(\xi, \eta)$, for $p = 5$ and $\alpha = 0.05$. This is an even function of $\xi$.

The minimum value of $\mathsf{SEL}(\xi, \eta)$ is 0.858, which is achieved at $(\xi, \eta) = (0, 0)$ i.e. when the treatment means are equal.

This plot provides a numerical illustration of the two theorems.



## Overall conclusion

1. The Yu and Hoff confidence intervals each **individually** has the desired coverage probability $1 - \alpha$.

   There is no statement about the joint coverage probability of these confidence intervals.

2. The scaled expected length SEL **diverges to infinity in some parts of the parameter space**

3. The results also suggest that if the treatment population means $\theta_1, \ldots, \theta_p$ are such that only one of these is an outlier then the Yu and Hoff confidence interval for the outlying treatment population mean will have a **very large expected length**, while the Yu and Hoff confidence intervals for the remaining treatment population means will be close to their usual confidence intervals